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MINIMUM MATRIX REPRESENTATION OF CLOSURE OPERATIONS

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Let a be a column of the $m \times n$ matrix M and A a set of its columns. We say that A implies a iff M contains no two rows equal in A but different in a . It is easy to see that if $\mathcal{I}_M(A)$ denotes the columns implied by A , then $\mathcal{I}_M(A)$ is a closure operation. We say that M represents this closure operation. $s(\mathcal{I})$ is the minimum number of the rows of the matrices representing a given closure operation. $s(\mathcal{I})$ is determined for some particular closure operations.

1. Introduction

A simple model of a data base [4] is a matrix. A row contains the data of one individual. A column contains the data of the same sort (e.g. date of birth). Let X denote the set of columns. Choose a subset $A \subseteq X$ and suppose that the data of an individual are known in the columns belonging to A . The individual (or row) is not necessarily determined, there can be more individuals (rows) having these data in the columns belonging to A . However all these individuals (rows) might agree in a column $b \notin A$. We say that b belongs to the closure $\mathcal{I}(A)$ of A if this happens for b with any choice of data in the columns belonging to A . We will see that this function \mathcal{I} mapping 2^X into 2^X satisfies (2)–(4). Such a function is called a closure operation. Conversely, if a closure operation \mathcal{I} is given, one can find a matrix generating exactly this closure operation in the above defined way [1, 5, 6]. Let $s(\mathcal{I})$ denote the minimum cardinality of rows of such matrices.

The main aim of the paper is to investigate the function $s(\mathcal{I})$. There are three kinds of results. In Section 3 we more or less determine $s(\mathcal{I})$ for some very special closure operations: the closure of any set of cardinality $\geq k$ is X while the closure of smaller sets A is A . Section 4 determines $s(\mathcal{I}_1 \times \mathcal{I}_2)$ in terms of $s(\mathcal{I}_1)$ and $s(\mathcal{I}_2)$. Finally we quote a result producing an \mathcal{I} with large $s(\mathcal{I})$.

2. Definitions

Let M be a matrix of m rows and n columns. The set of columns will be denoted by X . If $A \subseteq X$, $a \in X$ and M contains no two rows equal in A but different in a then we say that A *implies* a . The *closure* of A is

$$(1) \quad \mathcal{L}_M(A) = \{a: a \in X, A \text{ implies } a\}.$$

It is easy to see that the following rules are valid for $\mathcal{L}_M = \mathcal{L}$:

$$(2) \quad A \subseteq \mathcal{L}(A),$$

$$(3) \quad A \subseteq B \Rightarrow \mathcal{L}(A) \subseteq \mathcal{L}(B),$$

$$(4) \quad \mathcal{L}(\mathcal{L}(A)) = \mathcal{L}(A).$$

A function $\mathcal{L}: 2^X \rightarrow 2^X$ is called a *closure operation* if it satisfies (2)–(4).

Conversely, if \mathcal{L} is an arbitrary closure operation on an n -element groundset X , then there is an $m \times n$ matrix M such that $\mathcal{L}_M = \mathcal{L}$ [1, 5, 6]. We say in this case that M *represents* \mathcal{L} . The definition of our main target is the following:

$$(5) \quad s(\mathcal{L}) = \min\{m: M \text{ is an } m \times n \text{ matrix, } \mathcal{L}_M = \mathcal{L}\}.$$

A closure operation determines an important class of subsets, the class of keys. K is a *key* in \mathcal{L} if $\mathcal{L}(K) = X$. $\mathcal{K} = \mathcal{K}(\mathcal{L})$ denotes the family of *minimal keys* (K is a minimal key if it is a key but no proper subset of K is a key). It is obvious that $K_1, K_2 \in \mathcal{K}$, $K_1 \neq K_2$ imply $K_1 \not\subseteq K_2$. Families of subsets satisfying this condition are called *Sperner-families*. Hence \mathcal{K} is a Sperner-family. We say that a matrix M *represents* a given Sperner-family \mathcal{K} if $\mathcal{K} = \mathcal{K}(\mathcal{L}_M)$ holds. The maximal non-keys are called *antikeys*. Their family is denoted by

$$\mathcal{K}^{-1} = \{A: \nexists B \in \mathcal{K}, B \subseteq A, A \text{ is maximal for this property}\}.$$

Lemma 1. M represents the Sperner-family \mathcal{K} iff for any $A \in \mathcal{K}^{-1}$ M has two different rows having the same entries in the columns in A and any two rows equal in a $K \in \mathcal{K}$ are equal everywhere.

Proof. If M represents \mathcal{K} , then $\mathcal{K} = \mathcal{K}(\mathcal{L}_M)$ holds.

$K \in \mathcal{K}$ implies $\mathcal{L}_M(K) = X$, the second condition obviously follows. Similarly, $A \in \mathcal{K}^{-1}$ implies $\mathcal{L}_M(A) \neq X$ and hence we obtain the first condition.

Conversely, if both conditions are satisfied for M and \mathcal{K} , then (i) $\mathcal{L}_M(A) \neq X$ holds for any $A \in \mathcal{K}^{-1}$ and (ii) $\mathcal{L}_M(K) = X$ holds for any $K \in \mathcal{K}$.

(ii) and (3) imply that $\mathcal{L}_M(C) = X$ if $C \supseteq K$ for some $K \in \mathcal{K}$. Suppose now that C is not a superset of a member of \mathcal{K} . Then, by definition there is an $A \in \mathcal{K}^{-1}$ such that $C \subseteq A$. (i) and (3) imply $\mathcal{L}_M(C) \neq X$. That is, $\mathcal{L}_M(C) = X$ exactly for the supersets of members of \mathcal{K} : $\mathcal{K} = \mathcal{K}(\mathcal{L}_M)$. The proof is complete. \square

The following definition is an analogue of (5):

$$(6) \quad s(\mathcal{K}) = \min\{m: M \text{ is an } m \times n \text{ matrix representing } \mathcal{K}\}$$

where \mathcal{K} is a Sperner-family on an n -element set.

The k -uniform closure operation on an n -element groundset X is defined by

$$(7) \quad \mathcal{L}_k^n(A) = \begin{cases} X, & \text{if } |A| \geq k, \\ A, & \text{if } |A| < k. \end{cases}$$

The family of all k -element subsets of X is denoted by $\binom{X}{k}$. In general, there exist more than one closure operation with the same system \mathcal{K} of minimal keys ($\mathcal{K} = \mathcal{K}(\mathcal{L})$). The next Lemma states that if \mathcal{K} is the family of all k -element subsets of X , then \mathcal{L} is uniquely determined by $\mathcal{K} = \mathcal{K}(\mathcal{L})$.

Lemma 2. *Let any closure operation \mathcal{L} be defined on an n -element set X . Then*

$$\mathcal{K}(\mathcal{L}) = \binom{X}{k} \quad \text{iff} \quad \mathcal{L} = \mathcal{L}_k^n.$$

Proof. $\mathcal{K}(\mathcal{L}_k^n) = \binom{X}{k}$ is trivial. We have to verify the converse statement only. Suppose that $a \in \mathcal{L}(A) - A$ for some $A \subseteq X$ such that $|A| < k$. Then one can find a set B satisfying $|B| = k$, $B \supseteq A \cup \{a\}$. (2) implies $\mathcal{L}(B - a) \supseteq B - a$; (3) implies $\mathcal{L}(B - a) \supseteq \mathcal{L}(A) \ni a$. Hence $\mathcal{L}(B - a) \supseteq B$ follows. We obtain $\mathcal{L}(B - a) = \mathcal{L}(\mathcal{L}(B - a)) \supseteq \mathcal{L}(B) = X$ by (4) and (3). Consequently, there is a set $B - a$ of cardinality $< k$ with closure X . This contradiction shows that $a \in \mathcal{L}(A) - A$ cannot exist if $|A| < k$: $\mathcal{L}(A) = A$. $\mathcal{L}(A) = X$ for $|A| \geq k$ easily follows from $\mathcal{K}(\mathcal{L}) = \binom{X}{k}$ and (3). $\mathcal{L} = \mathcal{L}_k^n$ holds, the lemma is proved. \square

3. Minimum representation of uniform closure operations

First we repeat some results of [9]. We prove these statements for sake of completeness.

Lemma 3. *If an $m \times n$ matrix M represents \mathcal{K} , then*

$$\binom{m}{2} \geq |\mathcal{K}^{-1}|.$$

Proof. If $A \in \mathcal{K}^{-1}$, then by Lemma 1 there are different rows i, j such that they are equal in A . Take another member B of \mathcal{K}^{-1} . Let the corresponding pair of rows be i' and j' . If the unordered pairs $\{i, j\}$, $\{i', j'\}$ are equal, then these two different rows are equal in $A \cup B$. Consequently, $\mathcal{L}(A \cup B) \neq X$ and there is a $C \supseteq A \cup B$ with $C \in \mathcal{K}^{-1}$. By the definition of \mathcal{K}^{-1} this is possible only when $C = A$ and $C = B$, con-

tradicting our supposition $A \neq B$. Therefore to different members of \mathcal{K}^{-1} we have different pairs of rows satisfying the above condition. The number of pairs of rows of M must be $\geq |\mathcal{K}^{-1}|$. The lemma is proved. \square

Lemma 4.

$$(8) \quad \binom{s(\mathcal{L}_k^n)}{2} \geq \binom{n}{k-1}.$$

Proof. Let M be an $s(\mathcal{L}_k^n) \times n$ matrix representing \mathcal{L}_k^n . By Lemma 2, M also represents $\mathcal{K}(\mathcal{L}_k^n) = \binom{X}{k}$. It is easy to see that $\mathcal{K}^{-1}(\mathcal{L}_k^n) = \binom{X}{k-1}$. Then (8) follows by Lemma 3. \square

We will see that (8) gives a fairly good lower estimate on $s(\mathcal{L}_k^n)$. It is sharp for $k = 1, 2, n-1$. It seems to be sharp for $k = 3$ and $n \geq 7$. On the other hand it is sharp up to a constant depending on k for any fixed k when $n \rightarrow \infty$. The case $k = n$ needs another lemma. If M is an $m \times n$ matrix let $G(M)$ denote the graph whose vertices are the rows of M , two vertices are connected with an edge iff the set A of columns where the two rows are equal is non-empty. The edge is *labelled* by A .

Lemma 5. *Let M be a matrix and let A_1, \dots, A_r be the labels along a circuit of $G(M)$. Then*

$$(9) \quad \left(\bigcap_{i=1, i \neq j}^r A_i \right) - A_j = \emptyset \quad (1 \leq j \leq r).$$

Proof. Suppose that, on the contrary, (9) is non-empty, that is, there is a column, say the u th one, which is an element of all A_i but A_j . Let the vertices of the circuit be k_1, \dots, k_r in such a way that the edge (k_i, k_{i+1}) is labelled by A_i ($1 \leq i < r$) and (k_r, k_1) is labelled by A_r . From $u \in A_{j+1}$ it follows that the k_{j+1} st and k_{j+2} nd entries of the u th column are equal. The same holds for the k_{j+2} nd and k_{j+3} rd entries, etc. Consequently, the k_{j+1} st, k_{j+2} nd, \dots , k_r th, k_1 st, \dots , k_j th entries in the u th column are all equal. This leads to $u \in A_j$ contradicting the assumption. The proof is complete. \square

Theorem 1 [9].

$$\begin{aligned} s(\mathcal{L}_1^n) &= 2, & s(\mathcal{L}_2^n) &= \lceil (1 + \sqrt{1 + 8n})/2 \rceil, \\ s(\mathcal{L}_{n-1}^n) &= n, & s(\mathcal{L}_n^n) &= n + 1 \end{aligned}$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$.

Proof. By Lemma 2, $s(\mathcal{L}_k^n) = s(\mathcal{K}(\mathcal{L}_k^n)) = s(\binom{X}{k})$. We use the last form in the proof.

For $k=1$, (8) gives $s(\mathcal{U}_1^n) \geq 2$. The construction

$$\begin{array}{cccc} 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \end{array}$$

proves the equality.

For $k=2$, (8) gives

$$(10) \quad \binom{s(\mathcal{U}_2^n)}{2} \geq n.$$

Suppose that $s(\mathcal{U}_2^n)$ satisfies (10); we construct an $s(\mathcal{U}_2^n) \times n$ matrix M representing $\binom{X}{2}$. Any column of M contains exactly two zeros. Different columns contain different pairs of zeros. The other entries of the i th row are i ($1 \leq i \leq s(\mathcal{U}_2^n)$). It is easy to see, using Lemma 1, that M represents $\binom{X}{2}$. The least integer satisfying (10) can be expressed in the form given in the theorem.

For $k=n-1$, (8) gives $s(\mathcal{U}_{n-1}^n) \geq n$. The construction

$$\begin{array}{cccc} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 \end{array}$$

gives the equality.

Finally, suppose that the $m \times n$ matrix M realizes $\binom{X}{n} = \{X\}$. By Lemma 1 there is an edge in $G(M)$ labelled with A for any $(n-1)$ -element subset of X . $G(M)$ has n different edges of this kind. These edges cannot form a circuit because the $(n-1)$ -element subsets cannot satisfy (9), the lemma is applicable. $G(M)$ has at least $n+1$ vertices: $s(\mathcal{U}_n^n) \geq n+1$. The following construction gives equality

$$\begin{array}{cccc} 0 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1. \end{array}$$

The proof is complete. \square

Substituting $k=3$ into (8) we obtain

$$s(\mathcal{U}_3^n) \geq n.$$

$s(\mathcal{U}_3^3) = 4 > 3$ is proved and $s(\mathcal{U}_3^6) > 6$ can be verified by checking all the cases. We conjecture that the above inequality is sharp for all other cases:

Conjecture 1. $s(\mathcal{U}_3^n) = n$ for $n \geq 7$.

We are able to reduce this conjecture in the case $n = 3r + 1$ for another conjecture concerning a certain kind of resolvable Steiner triple systems:

Conjecture 2. *There is a system of 3-element subsets of an $n (= 3r + 1)$ -element set $\{1, 2, \dots, n\}$ satisfying the following conditions:*

- (1) *Any pair of elements is contained in exactly two 3-sets.*
- (2) *The family of 3-sets can be divided into n subfamilies where the i th subfamily is a partition of $\{1, 2, \dots, n\} - \{i\}$.*
- (3) *Exactly one pair of members of two different subfamilies meet in 2 elements.*

We show the construction of an $n \times n$ matrix M representing \mathcal{U}_3^n ($n = 3r + 1$) if the family in Conjecture 2 exists. We write zeros in the main diagonal. The i th row j th entry will be 1 if i is an element of the l th triple in the j th sub-family. It follows by condition (3) that for any two columns of M there are two rows equal in these columns. The rows are, of course, different due to the zeros. The first condition of Lemma 1 is satisfied. Condition (1) implies that any two rows agree in exactly two entries. Hence there are no two rows equal in any given triple of columns. The second condition of Lemma 1 is also satisfied. M represents \mathcal{U}_3^n , indeed.

Conjecture 2 follows for $n \equiv 1$ or $4 \pmod{12}$ from the following result of Hanani [11, 12]. There exists a Steiner system $S(4, 2, n)$ for these n 's. (I.e. we have a 4-uniform subsystem \mathcal{U} on n -element set V such that for every two $v_1, v_2 \in V$ there exists exactly one member $S \in \mathcal{U}$ such that $\{v_1, v_2\} \subset S$.) Consider the 4-uniform set-system \mathcal{U} over $\{1, 2, \dots, n\}$ and replace every member $S \in \mathcal{U}$ with 4 3-element subsets. The obtained set-system \mathcal{F} meets the condition of Conjecture 2, where the i th subfamily $\mathcal{F}_i = \{S - \{i\} : i \in S \in \mathcal{U}\}$.

We have proved the following.

Theorem 2.

$$(11) \quad s(\mathcal{U}_3^n) \geq n,$$

$$(12) \quad s(\mathcal{U}_3^n) = n \quad \text{for } n = 12k + 1 \text{ and } n = 12k + 4.$$

Corollary 1. $n \leq s(\mathcal{U}_3^n) \leq n + 8$.

Proof. It follows from Theorem 2 and the inequality $s(\mathcal{U}_3^n) \leq s(\mathcal{U}_3^{n+1})$. \square

Remark. In Theorem 2 we could have proved the following stronger statement: For $n \equiv 1$ or $4 \pmod{12}$ we have a partition G_1, G_2, \dots, G_n of the edge-set of complete directed graph \vec{K}_n ($\vec{K}_n = \{\langle u, v \rangle : 1 \leq u \neq v \leq n\}$, so it has $n(n-1)$ oriented edges) such that G_i consists of $(n-1)/3$ pairwise vertex-disjoint oriented triangles on the points $\{1, 2, \dots, n\} - \{i\}$ and $G_i \cup G_j$ ($i \neq j$) contains exactly one pair of oppositely oriented edges.

Conjecture 2'. *The above-mentioned statement about the complete directed graph \vec{K}_n holds for every $n \equiv 1 \pmod{3}$.*

This conjecture is much stronger than the usual statements about resolvable block-designs (cf. Hanani [12]).

Let us show the constructions for $n=4$ and 7:

$$\begin{array}{cccc}
 0 & 0 & 0 & 1 \\
 0 & 0 & 1 & 0 \\
 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & 0
 \end{array}
 \begin{array}{cccc}
 0 & 2 & 2 & 2 \\
 1 & 0 & 2 & 1 \\
 1 & 1 & 0 & 2 \\
 1 & 2 & 1 & 0 \\
 2 & 1 & 2 & 1 \\
 2 & 1 & 1 & 2 \\
 2 & 2 & 1 & 1
 \end{array}
 \begin{array}{cccc}
 1 & 1 & 1 & 1 \\
 2 & 2 & 1 & 2 \\
 1 & 2 & 2 & 1 \\
 2 & 1 & 2 & 1 \\
 0 & 1 & 2 & 0 \\
 2 & 0 & 1 & 2 \\
 1 & 2 & 0 & 1
 \end{array}$$

Theorem 3 [10].

$$(13) \quad \sqrt{2} \left(\frac{1}{k-1} \right)^{(k-1)/2} n^{(k-1)/2} < s(\mathcal{L}_k^n) < 2^{3k/2} n^{(k-1)/2} \quad (2 \leq k < n).$$

Proof. The left-hand side of (13) is an easy consequence of (8). We now give a construction proving the right-hand side.

Let p be a prime number. We show that there is a set D of cardinality $2 \lfloor \sqrt{p} \rfloor$ such that any integer satisfies

$$(14) \quad i \equiv d_1 - d_2 \pmod{p}$$

for some elements d_1, d_2 of D . The set D is defined by

$$D = \{0, 1, 2, \dots, a-1, 2a, 3a, \dots, (a-1)a\}$$

where $a = \lceil \sqrt{p} \rceil$. Suppose that i satisfies $0 \leq i < p$ and $i = al + r$ ($0 \leq r < a$). If $1 \leq l \leq a-2$ and $0 < r < a$, then $d_1 = (l+1)a$ and $d_2 = a-r$ satisfy (14). If $i = al$ ($2 \leq l \leq a-1$), then $d_1 = al$ and $d_2 = 0$ are suitable. $a = 3a - 2a$ and the rest can be expressed as a difference of zero and one of the numbers $1, 2, \dots, a-1$. (We used $3 \leq a-1$. Otherwise we have $p \leq 9$. These cases can be checked separately.)

As regards the cardinality of D we have

$$|D| = 2a - 2 = 2(\lceil \sqrt{p} \rceil - 1) = 2 \lfloor \sqrt{p} \rfloor.$$

Let \mathcal{P} be defined in the following way:

$$\mathcal{P} = \{c_{k-1}x^{k-1} + c_{k-2}x^{k-2} + \dots + c_1x + c_0 : c_0, \dots, c_{k-1} \in D, c_{k-1} = 0 \text{ or } 1\}.$$

Note that

$$|\mathcal{P}| = 2^k \lfloor \sqrt{p} \rfloor^{k-1}.$$

Let M be a $|\mathcal{P}| \times p$ matrix. Its rows are associated with members of \mathcal{P} . The j th entry of the row associated with $z(x) \in \mathcal{P}$ is $z(j) \pmod{p}$ ($0 \leq j \leq p-1$, $0 \leq z(j) \leq p-1$). We prove now that M represents \mathcal{L}_k^p . It is sufficient to prove, by Lemma 2, that M

represents $\binom{X}{k}$ (where $|X|=p$). Here we may use Lemma 1; we have to verify its conditions with $\mathcal{X}=\binom{X}{k}$, $\mathcal{X}^{-1}=\binom{X}{k-1}$ only.

Suppose that the rows associated with $z_1(x)$ and $z_2(x)$ have k equal entries:

$$z_1(t_i) \equiv z_2(t_i) \pmod{p} \quad (0 \leq t_1 < \dots < t_k < p).$$

Then the polynomial $z_1(x) - z_2(x)$ of degree $\leq k-1$ has k different roots. This contradiction proves that z_1 and z_2 are the same, the 'two' rows are only one.

Choose now the integers $0 \leq t_1 < \dots < t_{k-1} < p$ arbitrarily. We have to find two different rows with equal entries in the t_1 st, t_2 nd, ..., t_{k-1} st places. Consider the polynomial

$$w(x) = (x-t_1)(x-t_2)\dots(x-t_{k-1}) = x^{k-1} + a_{k-2}x^{k-2} + \dots + a_1x + a_0.$$

To a_i ($0 \leq i \leq k-2$) we can find two elements c_i and c'_i of D such that $a_i \equiv c_i - c'_i \pmod{p}$. Then $w(x) = z(x) - z'(x)$ holds where

$$z(x) = x^{k-1} + c_{k-2}x^{k-2} + \dots + c_1x + c_0 \quad \text{and}$$

$$z'(x) = c'_{k-2}x^{k-2} + \dots + c'_1x + c'_0.$$

$z(x)$ and $z'(x)$ are obviously different elements of \mathcal{P} . On the other hand, $z(t_i) \equiv z'(t_i) \pmod{p}$ holds, indeed. Both conditions of Lemma 1 are verified. M represents \mathcal{Z}_k^p , indeed. This proves

$$s(\mathcal{Z}_k^p) \leq 2^k p^{(k-1)/2}.$$

For arbitrary n we choose a prime number p satisfying $n \leq p \leq 2n$. It exists by Chebyshev's theorem. Then we construct a matrix representing \mathcal{Z}_k^p and omit $p-n$ columns. The matrix represents \mathcal{Z}_k^n . Hence

$$s(\mathcal{Z}_k^n) \leq 2^k p^{(k-1)/2} \leq 2^k (2n)^{(k-1)/2} \leq 2^{3k/2} n^{(k-1)/2}.$$

The theorem is proved. \square

The method of Theorem 3 gives a good estimate only for small k . For instance, for $k=n/2$ a much better estimate is known. It is proved in [6] that

$$(15) \quad s(\mathcal{X}) \leq |\mathcal{X}^{-1}| + 1$$

holds for any Sperner-family. The following matrix proves it. Let the 0th row consist of zeros while the i th ($1 \leq i \leq |\mathcal{X}^{-1}|$) row contains zeros and i 's: zeros in the column corresponding to the elements of the i th member of \mathcal{X}^{-1} . By (15),

$$s(\mathcal{Z}_{n/2}^n) = s\left(\begin{pmatrix} X \\ n/2 \end{pmatrix}\right) \leq \binom{n}{n/2} + 1 = 2^{n+o(n)}$$

follows. Our feeling is that the truth is closer to the lower estimate given by (8):

Conjecture 3. $\log_2 s(\mathcal{Z}_{n/2}^n) = n/2 + o(n)$.

For comparison let us quote another related result [7, 8] stating

$$s(\mathcal{L}) \leq (1 + o(1)) \binom{n}{n/2}$$

for any closure operation \mathcal{L} on n elements.

4. Direct products

Let \mathcal{L}_1 and \mathcal{L}_2 be closure operations on the disjoint ground sets X_1 and X_2 , resp. The *direct product* $\mathcal{L}_1 \times \mathcal{L}_2$ is defined by

$$(\mathcal{L}_1 \times \mathcal{L}_2)(A) = \mathcal{L}_1(A \cap X_1) \cup \mathcal{L}_2(A \cap X_2), \quad A \subseteq X_1 \cup X_2.$$

We prove the following, perhaps surprising

Theorem 4. $s(\mathcal{L}_1 \times \mathcal{L}_2) = s(\mathcal{L}_1) + s(\mathcal{L}_2) - 1$.

Proof. (1) Let us first prove the inequality

$$(16) \quad s(\mathcal{L}_1 \times \mathcal{L}_2) \leq s(\mathcal{L}_1) + s(\mathcal{L}_2) - 1$$

with a construction. Let the $s(\mathcal{L}_1) \times n_1$ matrix M_1 and the $s(\mathcal{L}_2) \times n_2$ matrix M_2 represent \mathcal{L}_1 and \mathcal{L}_2 , resp. We denote by α the last row of M_1 and by β the first row of M_2 . The matrix M is constructed in Fig. 1.

X_1	X_2
M_1 α	β \vdots β
	β
α \vdots α	M_2

Fig. 1.

M is an $(s(\mathcal{L}_1) + s(\mathcal{L}_2) - 1) \times (n_1 + n_2)$ matrix. We have to show that it represents $\mathcal{L}_1 \times \mathcal{L}_2$, that is,

$$a \in \mathcal{L}_M(A) \Leftrightarrow a \in \mathcal{L}_1(A \cap X_1) \cup \mathcal{L}_2(A \cap X_2).$$

We may assume $a \in X_1$ because of the symmetry. Then the above condition can be divided into two implications:

$$(17) \quad a \in \mathcal{L}_1(A \cap X_1) \Rightarrow \text{any two rows of } M \text{ equal in } A \text{ are equal in } a,$$

$$(18) \quad a \notin \mathcal{L}_1(A \cap X_1) \Rightarrow M \text{ has two rows equal in } A \text{ but different in } a.$$

To prove (17) suppose that $a \in \mathcal{L}_1(A \cap X_1)$ choose two rows of M with equal entries in A . If both of them start with a , they are equal in a . If one of them does not start with a then the first parts of these rows are two different rows of M_1 . By the definition of M_1 , if they are equal in $A \cap X_1$ then they are equal in a .

To prove (18) suppose that $a \notin \mathcal{L}_1(A \cap X_1)$. M_1 contains two rows equal in $A \cap X_1$ but different in a . The extensions of these rows in M satisfy the right hand side of (18).

M represents $\mathcal{L}_1 \times \mathcal{L}_2$, consequently (16) is proved.

(2) With the help of two lemmas we prove now the inequality

$$(19) \quad s(\mathcal{L}_1 \times \mathcal{L}_2) \geq s(\mathcal{L}_1) + s(\mathcal{L}_2) - 1.$$

Let M be a matrix representing $\mathcal{L}_1 \times \mathcal{L}_2$ and suppose that the first n_1 columns correspond to the groundset X_1 of \mathcal{L}_1 and the remaining n_2 columns correspond to the groundset X_2 of \mathcal{L}_2 . We want to prove that the number of rows of M is at least $s(\mathcal{L}_1) + s(\mathcal{L}_2) - 1$. The submatrix determined by the first n_1 columns in M is denoted by M_1 . The rest is denoted by M_2 .

Lemma 6. $\mathcal{L}_{M_2} = \mathcal{L}_2$.

Proof. Suppose that $A \subseteq X_2$, $a \in X_2$ and $a \in \mathcal{L}_2(A)$. Hence $a \in (\mathcal{L}_1 \times \mathcal{L}_2)(A)$ follows. If two rows of M are equal in A then, by the definition of M , they are also equal in a . Of course, this remains true if we consider the submatrix M_2 only. That is, we have proved $a \in \mathcal{L}_{M_2}(A)$. Conversely, suppose now $A \subseteq X_2$, $a \in X_2$ and $a \notin \mathcal{L}_2(A)$. Since $a \notin (\mathcal{L}_1 \times \mathcal{L}_2)(A)$ follows, M has two rows equal in A and different in a . These two rows in M_2 prove $a \notin \mathcal{L}_{M_2}(A)$. The proof is complete. \square

$\mathcal{L}_{M_1} = \mathcal{L}_1$ follows analogously. However, we need a somewhat stronger statement for M_1 :

Lemma 7. *Let N be a matrix. Suppose that the set of rows of N can be partitioned into k classes such that whenever $a \notin \mathcal{L}_N(A)$ holds, then there are two rows in one class which are equal in A and different in a . Then*

$$(20) \quad (\text{number of rows of } N) \geq s(\mathcal{L}_N) + k - 1.$$

Proof. We use induction on k . For $k = 1$, (20) follows by the definitions of $s(\mathcal{L}_N)$. Suppose now that N is partitioned into $k \geq 2$ classes satisfying the conditions of the lemma and that the statement is proved for smaller values.

\mathcal{L}_N depends only on the relationship of the entries in N : which ones are equal, which ones are different. Therefore we may suppose that N contains only positive integers.

If N has one or more columns with the same entry everywhere, then delete these columns and denote the new matrix by N_1 . The partition of rows of N is a ‘good’

partition in N_1 , too. On the other hand

$$(21) \quad s(\mathcal{L}_{N_1}) = s(\mathcal{L}_N)$$

obviously holds. Moreover $\mathcal{L}_{N_1}(\emptyset) = \emptyset$.

Let p_1, \dots, p_k be different prime numbers greater than any entry of N_1 . Multiply all entries in the i th class of rows by p_i ($1 \leq i \leq k$). The new matrix is denoted by N_2 . It is easy to see that

$$(22) \quad \mathcal{L}_{N_2} = \mathcal{L}_{N_1}$$

(since $\mathcal{L}_{N_1}(\emptyset) = \emptyset$) and N_2 contains no equal entries in different classes of rows.

Let $\gamma = (\gamma_1, \dots, \gamma_u)$ and $\delta = (\delta_1, \dots, \delta_u)$ be one of the rows of the first and second classes in N_2 , resp. We now delete γ and change any γ_i for δ_i in the i th column (for all i , $1 \leq i \leq u$). The new matrix is denoted by N_3 . The number of its rows is equal to the number of rows of N_2 minus 1. Let us prove that

$$(23) \quad \mathcal{L}_{N_3} = \mathcal{L}_{N_2}.$$

Suppose first that $a \in \mathcal{L}_{N_2}(A)$ and choose two rows, μ_3 and ν_3 of N_3 equal in A . The corresponding rows in N_2 are denoted by μ_2 and ν_2 , resp. If μ_2 and ν_2 are in the same class but not in the first one in N_2 , then $\mu_2 = \mu_3$, $\nu_2 = \nu_3$. Therefore μ_2 and ν_2 are equal in A ; $a \in \mathcal{L}_{N_2}(A)$ implies that they are equal in a . The same holds for μ_3 and ν_3 . $a \in \mathcal{L}_{N_3}(A)$ is proved. If μ_2 and ν_2 are both in the first class, then μ_2 and μ_3 differ only in the sense that γ_i is changed for δ_i everywhere. The same holds for ν_2 and ν_3 . It follows that μ_2 and ν_2 are equal in A , consequently in a . We obtain that μ_3 and ν_3 are also equal in a : $a \in \mathcal{L}_{N_3}(A)$. The last case is when μ_2 and ν_2 are in different classes. The supposition that μ_3 and ν_3 are equal in A implies either $A = \emptyset$ or that μ_2 and ν_2 are in the first and second classes, resp. $A = \emptyset$ is excluded by $\mathcal{L}_{N_2}(\emptyset) = \emptyset$. We may conclude that μ_2 is in the first class, ν_2 in the second one and they are different from γ and δ . $\nu_2 = \nu_3$ is obvious. Since μ_3 and ν_3 are equal in A they both must contain δ_i in the i th place if $i \in A$. Then $\nu_2 = \nu_3$ and δ are equal in A , consequently they are also equal in a . Their common entry here is δ_a . If μ_3 contains δ_i in the i th place when $i \in A$, then μ_2 contains γ_i there. Consequently, μ_2 and γ are equal in A and hence they are equal in a . Their common entry here is γ_a . We obtain that μ_3 contains δ_a in this column. Hence μ_3 and ν_3 are equal in a : $a \in \mathcal{L}_{N_3}(A)$.

Suppose now that $a \notin \mathcal{L}_{N_2}(A)$. N_2 contains two rows equal in A and different in a . If $A \neq \emptyset$, then the two rows are in the same class, consequently the corresponding rows in N_3 are also equal in A and different in a . $a \in \mathcal{L}_{N_3}(A)$ follows. If $A = \emptyset$, $a \in \mathcal{L}_{N_3}(\emptyset)$ would mean that there is a column with equal entries. This is impossible for $k \geq 3$. It is possible for $k = 2$ only when N_2 contains merely γ_a and δ_a in the column corresponding to a . However in this case we are not able to find two rows in one class satisfying the conditions of the lemma for $a \notin \mathcal{L}_{N_2}(\emptyset)$. This contradiction proves $a \notin \mathcal{L}_{N_3}(A)$ and (23).

Moreover the conditions of the lemma are satisfied with at least $k - 1$ classes for

N_3 . Therefore we may use the induction hypothesis:

$$(\text{number of rows of } N_3) \geq s(\mathcal{L}_{N_3}) + k - 2.$$

Hence we obtain

$$(\text{number of rows of } N) \geq s(\mathcal{L}_N) + k - 1$$

by (21), (22) and (23). The lemma is proved. \square

Let us turn back to the proof of the theorem, that is, more exactly, of (19). Form a partition of the rows of M_1 putting two rows in one class if their extension in M_2 is equal. Our aim is to apply Lemma 7 for M_1 . We know that $\mathcal{L}_{M_1} = \mathcal{L}_1$ by Lemma 6. Choose a and A so that $a \notin \mathcal{L}_{M_1}(A) = \mathcal{L}_1(A)$. Then $a \notin \mathcal{L}_1(A) \cup X_2 = (\mathcal{L}_1 \times \mathcal{L}_2)(A \cup X_2)$ holds and M contains two rows equal in $A \cup X_2$ but different in a . That is, there are two rows of M_1 equal in A , different in a and being in the same class of the partition. We may apply Lemma 7 for M_1 :

$$(24) \quad (\text{number of rows of } M_1) \geq s(\mathcal{L}_1) + (\text{number of different rows of } M_2) - 1.$$

Using Lemma 6, again, we obtain

$$(25) \quad (\text{number of different rows of } M_2) \geq s(\mathcal{L}_2).$$

(24) and (25) result in

$$(\text{number of rows of } M) \geq s(\mathcal{L}_1) + s(\mathcal{L}_2) - 1.$$

This proves (19) and the theorem. \square

The analogous question for Sperner-families as minimal keys is not really answered. If \mathcal{N}_1 and \mathcal{N}_2 are Sperner-families on the disjoint sets X_1 and X_2 , resp., then $\mathcal{N}_1 \times \mathcal{N}_2$ is defined as the family $\{A \cup B : A \in \mathcal{N}_1\}$ of subsets of $X_1 \cup X_2$. The proof of (16) works also here:

Theorem 5. $s(\mathcal{N}_1 \times \mathcal{N}_2) \leq s(\mathcal{N}_1) + s(\mathcal{N}_2) - 1$.

We found equality in many particular cases but it is not true in general, as the following example shows: let $X_1 = \{1, 2, 3, 4, 5\}$, $X_2 = \{6, 7, 8, 9, 10\}$, $\mathcal{N}_1 = \{\{1, 2\}, \{3, 4\}, \{1, 5\}, \{2, 5\}, \{3, 5\}, \{4, 5\}\}$, $\mathcal{N}_2 = \{\{6, 7\}, \{8, 9\}, \{6, 10\}, \{7, 10\}, \{8, 10\}, \{9, 10\}\}$. We show first $s(\mathcal{N}_1) = s(\mathcal{N}_2) \geq 5$. It is easy to see that

$$\mathcal{N}_1^{-1} = \{\{5\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}.$$

Lemma 3 implies $s(\mathcal{N}_1) \geq 4$. Suppose that $s(\mathcal{N}_1) = 4$ and the matrix M realizes it. $G(M)$ (see Lemma 5) has 4 vertices and its 5 edges are labelled with the members of \mathcal{N}_1^{-1} . Distinguishing several cases one can see that Lemma 5 implies that the sixth edge is labelled with $\{1, 2\}$, $\{1, 2, 5\}$, $\{3, 4\}$ or $\{3, 4, 5\}$. This contradicts the supposition that the key-set of M is \mathcal{N}_1 . This proves $s(\mathcal{N}_1) \geq 5$. On the other hand, the

following matrix shows $s(\mathcal{K}_1 \times \mathcal{K}_2) \leq 8$:

$$\begin{array}{l} 0\ 0\ 0\ 0\ 0 : 0\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0\ 0 : 1\ 1\ 1\ 1\ 0 \\ 0\ 0\ 0\ 0\ 0 : 0\ 1\ 0\ 1\ 1 \\ 1\ 1\ 1\ 1\ 0 : 0\ 0\ 0\ 0\ 0 \\ 1\ 1\ 1\ 1\ 0 : 1\ 1\ 1\ 1\ 0 \\ 1\ 1\ 1\ 1\ 0 : 1\ 0\ 0\ 1\ 2 \\ 0\ 1\ 0\ 1\ 1 : 0\ 0\ 0\ 0\ 0 \\ 1\ 0\ 0\ 1\ 2 : 1\ 1\ 1\ 1\ 0 \end{array}$$

(non-trivial!).

5. Closure relations with large $s(\mathcal{L})$

In [7] and [8] it is proved that there is an \mathcal{L} satisfying

$$s(\mathcal{L}) \geq s(\mathcal{K}(\mathcal{L})) \geq \frac{1}{n^2} \binom{n}{\lfloor n/2 \rfloor}.$$

However the proof is non-constructive; we are not able to find one with $s(\mathcal{L})$ more than $\sqrt{2} \binom{n}{\lfloor n/2 \rfloor}^{1/2}$.

We pose here the analogous question for $\mathcal{K} \subseteq \binom{X}{k}$. Let

$$f_k(n) = \max\{s(\mathcal{K}) : \mathcal{K} \subseteq \binom{X}{k}, |X| = n\}.$$

Theorem 6.

$$f_k(n) \geq \sqrt{2} \binom{2k-2}{k-1}^{\lfloor n/(2k-2) \rfloor / 2}$$

Proof. Take a partition $X = X_1 \cup \dots \cup X_q \cup Y$, where $q = \lfloor n/(2k-2) \rfloor$ and $|X_i| = 2k-2$ ($1 \leq i \leq q$). \mathcal{K} is defined by $\mathcal{K} = \{K : |K| = k, K \subseteq X_i \text{ for some } i\}$. It is easy to see that

$$\mathcal{K}^{-1} = \{A : |A \cap X_i| = k-1 \text{ for all } i, |A \cap Y| = |Y|\}.$$

Hence

$$|\mathcal{K}^{-1}| = \binom{2k-2}{k-1}^{\lfloor n/(2k-2) \rfloor}$$

follows and the theorem can be obtained by Lemma 3. \square

It is easy to see that $f_1(n) = 2$. Theorem 6 gives $f_2(n) \geq \sqrt{2} 2^{\lfloor n/2 \rfloor / 2} > 2^{n/4}$. It is surprising that such a ‘simple’ construction can have a big $s(\mathcal{K})$. However, we do not know the correct order of magnitude of $f_2(n)$.

6. Open problems

Besides Conjectures 1–3 we would like to pose some other related problems:

Problem 1. Give sufficient conditions for equality in Theorem 5.

Problem 2. Give methods for lower estimates of $s(\mathcal{L})$ and $s(\mathcal{K})$ deeper than Lemmas 3, 4 and 5.

Problem 3. Determine $\max\{|\mathcal{K}^{-1}| : \mathcal{K} \subseteq \binom{X}{k}, |X| = n\}$.

If $k = 2$, then \mathcal{K} is a graph on n vertices. \mathcal{K}^{-1} is the family of all maximal vertex-sets containing no edge of this graph. The problem asks for what graph is this family the largest. Moon and Moser [13] solved this graph-theoretical question. Our problem is its analogue for hypergraphs.

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